Bond Market Completeness and Attainable Contingent Claims

Erik Taflin*†

May 2003, Version 2005.02.12

Abstract

A general class, introduced in [7], of continuous time bond markets driven by a standard cylindrical Brownian motion $\bar{W}$ in ℓ^2, is considered. We prove that there always exist non-hedgeable random variables in the space $\mathsf{D}_0 = \cap_{p\geq 1} L^p$ and that D_0 has a dense subset of attainable elements, if the volatility operator is non-degenerated a.e. Such results were proved in [1] and [2] in the case of a bond market driven by finite dimensional B.m. and marked point processes. We define certain smaller spaces D_s, $s > 0$ of European contingent claims, by requiring that the integrand in the martingale representation, with respect to $\bar{W}$, takes values in weighted ℓ^2 spaces $\ell^{s,2}$, with a power weight of degree s. For all $s > 0$, the space D_s is dense in D_0 and is independent of the particular bond price and volatility operator processes.

A simple condition in terms of $\ell^{s,2}$ norms is given on the volatility operator processes, which implies if satisfied, that every element in D_s is attainable. In this context a related problem of optimal portfolios of zero coupon bonds is solved for general utility functions and volatility operator processes, provided that the ℓ^2-valued market price of risk process has certain Malliavin differentiability properties.

Keywords: Complete markets, bond portfolios, utility optimization, Hilbert space valued processes, Malliavin calculus
JEL Classification: C61, C62, G10, G11
MSC: 91B28, 49J55, 60H07, 90C46, 46E35

*Permanent address: EISTI, Ecole International des Sciences du Traitement de l'Information, Avenue du Parc, 95011 Cergy, France; taflin@eisti.fr

†This article was partially prepared at CEREMADE, Université Paris IX - Dauphine, Place du Maréchal-de-Lattre-de-Tassigny, 75775 Paris (Cedex 16), France, taflin@ceremade.dauphine.fr

1 Introduction

In this paper we consider the problem of completeness of continuous time markets of zero-coupon bonds, with arbitrary positive time to maturity. To fix the ideas, contingent claims will be elements of the space $\mathsf{D}_0 = \cap_{p \geq 1} L^p$, where the L^p spaces are defined with respect to an apriori given probability measure P. Introducing the zero-coupon markets, we follow the Hilbert space construction given in reference [7], which permits a unified approach to bond and stock markets. The zero-coupon price p_t, at a given time t, is as a function of time to maturity an element of a certain Sobolev space H of continuous functions. The basic random object in the theory is the price curve $p_t \in H$. This is an ∞-dimensional object and the number of random-sources influencing its evolution is big (cf. end of the introduction of [2] and [4]). Since the dimension of H is countable infinite, it is natural that the price process p is driven by a countable infinite number of random sources. Therefore, in [7] the evolution of the price in H is given by a diffusion model driven by a countable infinite number of independent standard Brownian motions (Bm.), i.e. a standard cylindrical Brownian motion.

From the point of view of sources of randomness, the construction in reference [7] is complementary to and generalizes the bond market driven by a finite dimensional Bm. and marked point processes, with possible infinite mark space, introduced in [1] and [2]. A notable difference is that the price process in [1] and [2] takes values in a Banach space (a certain sup. normed subspace of $C([0, \infty[\,))$, which in a general approach requires a more sophisticated stochastic integration theory than for Hilbert space valued processes. The completeness of a bond market driven by an infinite number of random sources was studied in [1] and [2] for their jump-diffusion model. It was proved that such a market with an infinite mark space, is approximately complete (Th. 6.11 of [2]), i.e. the set of hedgeable claims is dense in a certain sense in the set of claims. However, in general this market is not complete (Prop. 4.7 of [1]). This was further developed in [6]. The basic reason for this complication, compared with stock markets, is that the martingale operator (cf. formula (6.8) of [2]), being the product of the discounted zero-coupon price and the volatility operator, is a.e. a compact operator defined on an ∞-dimensional topological vector space (TVS). In the case of the Hilbert space model in [7] the situation is similar. There the martingale operator is compact a.e. (see formula (5.3) and Remark 5.1 of [7]). The hedging operator, i.e. the adjoint of the martingale operator is then also compact a.e. Intuitively, the market can only be complete if the hedging operator is a.e. surjective which is never the case, for a countable infinity of Brownian motions.

The first purpose of this article is to establish rigorously that the bond market with a usual derivative market such as D_0 cannot be complete, in the case of a countable infinity of Brownian motions (Theorem 4.1). This is in strong contrast with the case of a finite number of random sources, where this market is complete when the volatility operator satisfies certain non-degeneracy conditions (cf. [7] formula (3.8) and Remark 5.3).

This raises naturally the question of how to generalize the usual concept of a complete market, tailored for finite dimensional markets, to bond markets. If the martingale operator has trivial kernel a.e. then D_0 has a dense subspace of hedgeable elements (Theorem 4.2). However, this does not give any information on what the subset of hedgeable elements is. Roughly this corresponds to an approximately complete market, introduced in the different context of [1] and [2]. The solution adapted in this article simply consists of restricting the set of contingent claims to an allowed subspace $A \subset \mathsf{D}_0$ which satisfies:

$$(i)\ \textit{A is a locally convex complete TVS} \text{ and } (ii)\ \textit{A is dense in } \mathsf{D}_0. \tag{1.1}$$

Condition (i) permits to study if it is possible to choose the hedging portfolio as a continuous function of the contingent claim. Condition (ii) implies that the price (if continuous on D_0) of each element in D_0 is determined by the price of elements in A. The bond market, is then said to be relatively complete with respect to the allowed set A of contingent claims or just *A-complete*, if all elements in A are attainable. The idea here is that it should be easy to check whether or not a contingent claim X is in A. If $X \in A$ then X is hedgeable by definition, while if $X \notin A$ then we can only conclude that there is a sequence, not necessarily bounded, of self-financing portfolios with terminal value converging to X. Since the portfolio sequence can be unbounded the approximation scheme is difficult to use in practice and one needs at least a measure of risk, which permits to pick the best "approximate portfolio" in the sequence.

The second purpose of the article is to introduce spaces D_s, $s \geq 0$ of allowed European contingent claims satisfying (1.1) and sufficiently large to contain all commonly used derivatives, including those with discontinuous pay-off functions. The main point in the definition of D_s, $s \geq 0$ is that the integrand in the stochastic integral representation of elements in D_s decreases uniformly at a rate given by weighted ℓ^2-spaces with norm $y \mapsto (\sum_{i\geq 1}(1+i^2)^s(y^i)^2)^{1/2}$. The spaces D_s, $s \geq 0$ are independent of the particular bond price and volatility operator processes and $\mathsf{D}_s \subset \mathsf{D}_{s'}$, for $s' \leq s$. The third purpose of the article is to give conditions on the volatility operator (Condition II) such that the market is D_s-complete for certain $s > 0$ (Theorem 4.3).

The forth purpose of the article is to apply the D_s-completeness of the market to the optimal portfolio problem considered in [7]. There, the optimal terminal discounted wealth $\hat{X}$ was first found (Th.3.3 of [7]) under general conditions and then a hedging portfolio $\hat{\theta}$ of $\hat{X}$ was constructed for certain cases (deterministic volatility Th.3.8; finite number of Bm. Th.3.6). $\hat{X}$ is not always hedgeable, but to cover more general situations where it is hedgeable, so an optimal portfolio $\hat{\theta}$ exists, we here impose

$$(iii)\ A \text{ is an algebra under pointwise multiplication.} \tag{1.2}$$

In fact, $\hat{X}$ is a C^1 function, polynomially bounded together with its derivative, of dQ/dP for a martingale measure Q. Knowing that some claim like $\ln(dQ/dP)$ is attainable, we use the algebraic properties of A to prove that $\hat{X}$ is also attainable. Now, D_s, $s > 0$ is not an algebra (Remark 3.4). However, we define a subspace $\mathsf{D}_s^1 \subset \mathsf{D}_s$ of once Malliavin differentiable contingent claims, which is seen to be an algebra by generalizing the use made of the Clark-Ocone representation formula in [7]. The D_s^1-completeness of the market leads to a fairly general solution of the optimal portfolio problem (4.9) (Theorem 4.5). Reference [13] studies the optimal portfolio problem, within (essentially) the framework of the jump-diffusion model of [2]. Existence of optimal terminal discounted wealth is established. However, the hedging problem is only studied in the sense of approximate hedging, so it does not establish the existence of an optimal portfolio.

We note (Remark 3.5) that the spaces D_s are more appropriate for the study of general hedging problems than D_s^1, since the latter do not contain non-Malliavin-differentiable claims, in particular not binary options. A Malliavin-Clark-Ocone formalism was also adapted recently in reference [4], for the construction of hedging portfolios in a Markovian context, with a Lipschitz continuous (in the bond price) martingale operator. This guaranties that the Malliavin derivative of the bond price is proportional to the martingale operator (formula (30) of [4]). Hedging is then achieved for a restricted class of claims, namely European claims being a Lipschitz continuous function in the price of the bond at maturity.

The main results are proved in §5 and auxiliary needed results, difficult to find on suitable form, are proved in Appendix A.

Acknowledgment: The author would like to thank Ivar Ekeland and Nizar Touzi for fruitful discussions and the anonymous referees for constructive suggestions.

2 Zero-coupon markets and portfolios

Following closely [7], we first introduce zero-coupon markets and portfolios. We consider a continuous time zero-coupon market, with some finite time horizon $\bar{T} > 0$. At any date $t \in \mathbb{T} = [0, \bar{T}]$, one can trade zero-coupon bonds with maturity $t + T$, where the time *to* maturity $T \in [0, \infty[$.

Uncertainty is modeled by a complete filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$, where $\mathcal{A} = \{\mathcal{F}_t \mid 0 \leq t \leq \bar{T}\}$, is a filtration of the σ-algebra $\mathcal{F} = \mathcal{F}_{\bar{T}}$. The random sources are given by independent Brownian motions W^i, $i \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N} - \{0\}$. The filtration $\mathcal{A}$ is generated by the W^i, $i \in \mathbb{N}^*$.

We denote by $p_t(T)$ the price at time t of a zero-coupon yielding one unit of account at time $t + T$, $t \in \mathbb{T}$, $T \geq 0$, so that $p_t(0) = 1$. For a zero-coupon price, which is a strictly positive C^1 function in the time to maturity, the instantaneous forward rate contracted at $t \in \mathbb{T}$ for time to maturity $T \geq 0$ is

$$f_t(T) = -\frac{1}{p_t(T)} \frac{\partial p_t(T)}{\partial T}, \tag{2.1}$$

in the Musiela parameterization, the spot interest rate at t is $r_t = f_t(0)$ and the discounted zero-coupon price at time t is $\bar{p}_t = p_t \exp(-\int_0^t r_\tau d\tau)$.

We introduce Hilbert spaces H and $\tilde{H}$ of continuous real-valued functions, which will play the role of state spaces of the price process p and of drift and volatility processes respectively. Given $\tilde{s} \in]1/2, 1[$, let H be the subspace of all $f \in L^2([0, \infty[\,)$ satisfying

$$\int_{x \geq 0} |f(x)|^2 \, dx + \int_{x,\, y \geq 0} |f(x) - f(y)|^2 |x - y|^{-1-2\tilde{s}} \, dx \, dy < \infty. \tag{2.2}$$

H is a Sobolev space, which we now give its usual Hilbert space structure, often more easy to use than the one defined by the equivalent norm given by (2.2). For $s \in \mathbb{R}$, let H^s (cf. §7.9 of [9]) be the usual Sobolev space of real tempered distributions f on $\mathbb{R}$ such that the function $x \mapsto (1 + |x|^2)^{s/2} \hat{f}(x)$ is an element of $L^2(\mathbb{R})$, where $\hat{f}$ is the Fourier transform[1] of f, endowed with the norm:

$$\|f\|_{H^s} = (\int (1 + |x|^2)^s \, |\hat{f}(x)|^2 dx)^{1/2}.$$

We note that by Plancherel's Theorem (cf. §2, Ch. VI of [16]) $H^0 = L^2(\mathbb{R})$. The dual $(H^s)'$ of H^s is identified with H^{-s} by the continuous bilinear form $< , >: H^{-s} \times H^s \mapsto \mathbb{R}$:

$$< f \, , \, g >= \int \overline{\hat{f}(x)} \; \hat{g}(x) dx, \tag{2.3}$$

[1] In $\mathbb{R}^n$ we denote $x \cdot y = \sum_{1 \leq i \leq n} x_i y_i$, $x, y \in \mathbb{R}^n$ and we define the Fourier transform $\hat{f}$ of f by $\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-iy \cdot x) f(x) dx$.

where $\overline{z}$ is the complex conjugate of z. If $s > 1/2$ and $f, g \in H^s$, then f is Hölder continuous of order $s - 1/2$, $f(x) \to 0$, as $|x| \to \infty$ and there exists a constant C independent of f and g such that $\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}$ (cf. [3] and §7.9 of [9]). In particular if $s > 1/2$, then $H^s \subset C^0 \cap L^\infty$. We remind that $\tilde{s} > 1/2$. In $H^{\tilde{s}}$, consider the set $H_-^{\tilde{s}}$ of functions with support in $]-\infty, 0]$. It is a closed subspace of $H^{\tilde{s}}$, so the quotient space

$$H = H^{\tilde{s}}/H_-^{\tilde{s}} \tag{2.4}$$

is a Hilbert space as well. It follows (cf. formula (7.9.4) of [9]) that the norm defined by (2.2) is equivalent to $\| \ \|_H$. To sum up, a real-valued function f on $[0, \infty[$ belongs to H if and only if it is the restriction to $[0, \infty[$ of some function in $H^{\tilde{s}}$ and that H', the dual of H, is the set of all distributions in $H^{-\tilde{s}}$ with support in $[0, \infty[$. In particular, H inherits from $H^{\tilde{s}}$ the property of being a Banach algebra and H' contains all bounded Radon measures with support in $[0, \infty[$.

To motivate the introduction of $\tilde{H}$ suppose that $F \in \tilde{H}$ is the value of lets say a volatility process. We then impose that $Fg \in H$ for all $g \in H$. As we saw, if $F \in H$, then this condition is satisfied. It is also the case for functions F on $[0, \infty[$, such that $F = a + f$, for some $a \in \mathbb{R}$ and $f \in H$. Such F permits to consider volatilities, which do not go to zero when the time to maturity goes to infinity. We here choose $\tilde{H}$ to be functions of this form, even if more general spaces are possible. Then the Hilbert space $\tilde{H} = \mathbb{R} \oplus H$, since the decomposition of $F = a + f$, $a \in \mathbb{R}$ and $f \in H$ is unique. The norm is given by

$$\|F\|_{\tilde{H}} = (a^2 + \|f\|_H^2)^{1/2}. \tag{2.5}$$

The dual $\tilde{H}'$, of $\tilde{H}$ is identified with $\mathbb{R} \oplus H'$ by extending the bi-linear form, defined in (2.3), to $\tilde{H}' \times \tilde{H}$:

$$< F \, , \, G >= ab + < f \, , \, g >, \tag{2.6}$$

where $F = a + f \in \tilde{H}'$, $G = b + g \in \tilde{H}$, $a, b \in \mathbb{R}$, $f \in H'$ and $g \in H$.

In order to introduce the bond dynamics, let $\mathcal{L}$ denote the semigroup of left translations defined on real functions on $[0, \infty[$:

$$(\mathcal{L}_a f)(T) = f(a + T) \tag{2.7}$$

where $a \geq 0$, $T \geq 0$. $\mathcal{L}$ acts as a strongly continuous contraction semi-group in H (resp. $\tilde{H}$). The infinitesimal generator is denoted ∂ and its domain[2] $\mathcal{D}(\partial)$ is denoted H_1 (resp. $\tilde{H}_1$). The norm in H_1 (resp. $\tilde{H}_1$) is defined by

$$\|f\|_{H_1} = (\|f\|_H^2 + \|\partial f\|_H^2)^{1/2} \quad (\text{resp. } \|F\|_{\tilde{H}_1} = (\|F\|_{\tilde{H}} + \|\partial F\|_{\tilde{H}}^2)^{1/2}). \tag{2.8}$$

[2] $\mathcal{D}(A)$, $\mathcal{K}(A)$ and $\mathcal{R}(A)$ denote respectively the domain, the kernel and the range of a linear operator A. $B^\perp$ stands for the annihilator of a subset B of a TVS.

Throughout the paper, we shall assume that, $\bar{p}$ is a continuous strictly positive H_1-valued progressively measurable processes, with respect to $\mathcal{A}$, given by an equation of the HJM type (see [8] and equation (2.11) of [7])

$$\bar{p}_t = \mathcal{L}_t \bar{p}_0 + \int_0^t \mathcal{L}_{t-s}(m_s \bar{p}_s) ds + \int_0^t \sum_{i \in \mathbb{N}^*} \mathcal{L}_{t-s}(\sigma_s^i \bar{p}_s) dW_s^i, \tag{2.9}$$

with boundary condition

$$\bar{p}_t(0) = \exp(\int_0^t \frac{\partial \bar{p}_s(0)}{\bar{p}_s(0)} ds), \tag{2.10}$$

for $t \in \mathbb{T}$, where σ_t^i, $i \in \mathbb{N}^*$, and m_t are progressively measurable $\tilde{H}$-valued processes satisfying

$$\sigma_t^i(0) = 0 \text{ for } i \in \mathbb{N}^* \tag{2.11}$$

and

$$m_t(0) = 0. \tag{2.12}$$

Identifying[3] W with a ℓ^2 cylindrical Wiener process (c.f. §4.3.1 of reference [5]), equation (2.9) can be written on a more compact form. Let σ be the progressively measurable $L(\ell^2, \tilde{H})$-valued[4] process defined by $\sigma_t x = \sum_{i \in \mathbb{N}^*} \sigma_t^i x_i$. Then equation (2.9) reads (cf. equation (5.5) of [7])

$$\bar{p}_t = \mathcal{L}_t \bar{p}_0 + \int_0^t \mathcal{L}_{t-s} \bar{p}_s m_s ds + \int_0^t \mathcal{L}_{t-s} \bar{p}_s \sigma_s dW_s. \tag{2.13}$$

We shall assume that σ takes its values in the subspace of Hilbert-Schmidt operators of $L(\ell^2, \tilde{H})$, which permits to give a meaning to the stochastic integral in equation (2.13) (cf. §4.3.1 of [5]).

A portfolio is an H'-valued progressively measurable process θ defined on $\mathbb{T}$. If θ is a portfolio, then its discounted value at time t is

$$\bar{V}_t(\theta) = < \theta_t \,,\, \bar{p}_t > . \tag{2.14}$$

θ is an *admissible portfolio* if

$$\|\theta\|_{\mathsf{P}}^2 = E\left(\int_0^{\bar{T}} (\|\theta_t\|_{H'}^2 + \|\sigma_t' \theta_t \bar{p}_t\|_{\ell^2}^2) dt + (\int_0^{\bar{T}} |< \theta_t \,,\, \bar{p}_t m_t >| dt)^2\right) < \infty, \tag{2.15}$$

[3] ℓ^2 is the usual Hilbert space of all real sequences $x = (x_1, \ldots, x_n, \ldots)$, with norm $\|x\|_{\ell^2} = \sum_{n \geq 1} (x_n)^2$.

[4] $L(E, F)$ denotes the space of linear continuous mappings from E into F, $L(E) = L(E, E)$.

where $\sigma^{'}$ is the adjoint process of σ defined by $< f\,,\, \sigma_t x >= (\sigma^{'}_t f, x)_{\ell^2}$, for all $f \in \tilde{H}^{'}$ and $x \in \ell^2$. Explicitly we have:

$$\sigma^{'}_t f = (< f\,,\, \sigma^1_t >, \ldots, < f\,,\, \sigma^i_t >, \ldots). \tag{2.16}$$

The set of all admissible portfolios defines a Banach space P for the norm $\|\ \|_{\mathsf{P}}$. A portfolio is self-financing if

$$d\bar{V}_t(\theta) =< \theta_t\,,\, \bar{p}_t m_t > dt + \sum_{i\in\mathbb{N}^*} < \theta_t\,,\, \bar{p}_t \sigma^i_t > dW^i_t. \tag{2.17}$$

The subspace of all self-financing portfolios in P is a Banach space P_{sf}.

We next impose a condition on the zero-coupon market.

Condition I

a) The initial condition $\bar{p}_0$ satisfies:

$$\bar{p}_0 \in H_1,\ \bar{p}_0(0) = 1,\ \bar{p}_0 > 0, \tag{2.18}$$

b) σ^i, $i \in \mathbb{N}^*$ are given progressively measurable $\tilde{H}_1$-valued processes, such that (2.11) is satisfied and such that for all $a \in [1, \infty[$,

$$E((\int_0^{\bar{T}} \sum_{i\in\mathbb{N}^*} \|\sigma^i_t\|^2_{\tilde{H}_1} dt)^a + \exp(a \int_0^{\bar{T}} \sum_{i\in\mathbb{N}^*} \|\sigma^i_t\|^2_{\tilde{H}} dt)) < \infty, \tag{2.19}$$

c) There exists a family $\{\Gamma^i \mid i \in \mathbb{N}^*\}$ of real-valued progressively measurable processes such that

$$m_t + \sum_{i\in\mathbb{N}^*} \Gamma^i_t \sigma^i_t = 0, \tag{2.20}$$

and

$$E(\exp(a \int_0^{\bar{T}} \sum_{i\in\mathbb{N}^*} |\Gamma^i_t|^2 dt)) < \infty, \quad \forall a \geq 0. \tag{2.21}$$

This condition gives a mathematical meaning to and guarantees the existence of a solution to the mixed initial value and boundary value problem (2.9) and (2.10) (see Theorem 2.1 and Theorem 2.2 of [7]):

Theorem 2.1 *If Condition I is satisfied, then equation (2.9) has, in the set of continuous progressively measurable H-valued processes, a unique solution $\bar{p}$. This solution has the following properties: $\bar{p}$ is a strictly positive continuous progressively measurable H_1-valued processes and the boundary condition (2.10) is satisfied. Moreover, if $u \in [1, \infty[$ and $\bar{q}_t = \bar{p}_t / \mathcal{L}_t \bar{p}_0$ then $\bar{p} \in L^u(\Omega, P, L^\infty(\mathbb{T}, H_1))$, $\bar{q}, 1/\bar{q} \in L^u(\Omega, P, L^\infty(\mathbb{T}, \tilde{H}_1))$ and $\bar{p}(0), 1/\bar{p}(0) \in L^u(\Omega, P, L^\infty(\mathbb{T}, \mathbb{R}))$.*

Condition I also guarantees the existence of a martingale measure (see Theorem 2.8 and Corollary 2.10 of [7]). In order to state the result let

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \sum_{i\in\mathbb{N}^*}(\Gamma_s^i)^2 ds + \int_0^t \sum_{i\in\mathbb{N}^*}\Gamma_s^i dW_s^i\right), \tag{2.22}$$

where $t \in \mathbb{T}$.

Theorem 2.2 *If Condition I is satisfied, then ξ is a martingale with respect to $(P, \mathcal{A})$ and $\sup_{t\in\mathbb{T}} \xi_t^\alpha \in L^1(\Omega, P)$ for each $\alpha \in \mathbb{R}$. The measure Q, defined by*

$$dQ = \xi_{\bar{T}} dP,$$

is equivalent to P on $\mathcal{F}_{\bar{T}}$ and $t \mapsto \bar{W}_t^i = W_t^i - \int_0^t \Gamma_s^i ds$, $t \in \mathbb{T}$, $i \in \mathbb{N}^*$ are independent Wiener process with respect to $(Q, \mathcal{A})$. If moreover $\theta \in \mathsf{P}_{sf}$, then $\bar{V}(\theta)$ is a $(Q, \mathcal{A})$-martingale and $E(\sup_{t\in\mathbb{T}}(\bar{V}_t(\theta))^2) < \infty$.

Thus, when Condition I is satisfied, the self-financing criteria (2.17) is equivalent to

$$\bar{V}_t(\theta) = \bar{V}_0(\theta) + \int_0^t \sum_{i\in\mathbb{N}^*} < \theta_s\, , \, \bar{p}_s \sigma_s^i > d\bar{W}_s^i. \tag{2.23}$$

The expected value of a random variable X with respect to Q is denoted $E_Q(X)$ and $E_Q(X) = E(\xi_{\bar{T}} X)$.

3 Contingent claims

In this section we consider contingent claims $X \in L^p(\Omega, P, \mathcal{F})$, for all $p \geq 1$, i.e. $X \in \mathsf{D}_0 = \cap_{p\geq 1} L^p(\Omega, P, \mathcal{F})$ and we assume that Condition I is satisfied. This is a convenient space, since it contains most usually traded contingent claims and it gives an easy mathematical analysis. It has also a certain invariance with respect to the probability measure P, which we shall formulate in a slightly more general context. For a Banach space F we define the vector space $\mathsf{D}_0(F) = \cap_{1\leq p} L^p(\Omega, P, \mathcal{F}, F)$ and we denote $\mathsf{D}_0 = \mathsf{D}_0(\mathbb{R})$. It is endowed with the topology induced by the countable sequence of seminorms

$$X \mapsto \|X\|_{L^n(\Omega,P,\mathcal{F},F)}, \tag{3.1}$$

$n \in \mathbb{N}^*$. $\mathsf{D}_0(F)$ is then a Fréchet space. Replacing P by the martingale measure Q in the definition of D_0 gives the same space, so in the sequel of this section we shall use Q :

Lemma 3.1 *If Condition I is satisfied and if F is a Banach space, then*

$$\mathsf{D}_0(F) = \cap_{1 \le p} L^p(\Omega, Q, \mathcal{F}, F)$$

and the topology of $\mathsf{D}_0(F)$ is induced by the sequence of seminorms $X \mapsto \|X\|_{L^n(\Omega,Q,\mathcal{F},F)}$, $n \in \mathbb{N}^*$.

In general there are non-attainable random variables in the space D_0 (see Theorem 4.1). In order to obtain complete markets, we shall therefore restrict the set of allowed contingent claims. To specify various subspaces of D_0 of allowed contingent claims we introduce certain Hilbert spaces and the isomorphism of square integrable random variables and square integrable progressively measurable processes.

For $s \in \mathbb{R}$, let $\ell^{s,2}$ be the Hilbert space of real sequences endowed with the norm

$$\|x\|_{\ell^{s,2}} = (\sum_{i \in \mathbb{N}^*} (1+i^2)^s |x^i|^2)^{1/2}. \tag{3.2}$$

Obviously $\ell^2 = \ell^{0,2}$. The operator j in $\ell^{s,2}$, with domain $\ell^{s+1,2}$ and given by

$$(\mathsf{j}x)^k = kx^k \tag{3.3}$$

is selfadjoint and strictly positive. Obviously, if $a \ge 0$ then the domain of j^a is $\ell^{s+a,2}$ and for all $s_1, s_2 \ge 0$, $\mathsf{j}^{s_1+s_2} = \mathsf{j}^{s_1}\mathsf{j}^{s_2}$ (domains included).

Let ν be the product of the Lebesgue measure on $\mathbb{T}$ and the counting measure on $\mathbb{N}^*$ and let $L^2(\mathbb{T} \times \mathbb{N}^*)$ be the space of real square integrable functions with respect to ν. We have $L^2(\mathbb{T} \times \mathbb{N}^*) \cong L^2(\mathbb{T}, \ell^2)$. For $p \ge 2$, $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$ is the Banach space with norm defined by

$$\|x\|_{L^p(\Omega,Q,L^2(\mathbb{T},\ell^2))} = (E_Q(\int_0^{\bar{T}} \|x_t\|^2_{\ell^2} dt)^{p/2})^{1/p}. \tag{3.4}$$

L^p_a denotes the closed subspace of all progressively measurable elements (modulo equivalence) in $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$. For all $s \ge 0$ and $p \ge 2$, let $L^p_{a,s} = L^p_a \cap L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2}))$. It is a closed subspace of $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2}))$ and we give $L^p_{a,s}$ the corresponding Banach space structure. The operator $\mathfrak{J}$ in $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2}))$, $p \ge 2$ and $s \in \mathbb{R}$, defined by its domain $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s+1,2}))$ and the expression

$$(\mathfrak{J}x)_t(\omega) = \mathsf{j}x_t(\omega) \tag{3.5}$$

is a closed operator. For $a \ge 0$, the fractional power given by $(\mathfrak{J}^a x)_t(\omega) = \mathsf{j}^a x_t(\omega)$ and by its domain $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s+a,2}))$, is also a closed operator. By restriction $\mathfrak{J}$ defines an operator in $L^p_{a,s}$ with domain $L^p_{a,s+1}$, which we also denote by $\mathfrak{J}$. For $p = 2$, $\mathfrak{J}$ is selfadjoint and positive.

For completeness we state the following result, proved in Appendix A:

Lemma 3.2 *The operator U given by*

$$U(c,x) = c + \sum_{i\in\mathbb{N}^*} \int_0^{\bar{T}} x_t^i d\bar{W}_t^i. \tag{3.6}$$

is unitary on $\mathbb{R}\oplus L_a^2$ to $L^2(\Omega,Q,\mathcal{F})$. Let $U^* : L^2(\Omega,Q,\mathcal{F}) \to \mathbb{R}\oplus L_a^2$ be the adjoint of U. If $p\geq 2$, then $U(\mathbb{R}\oplus L_a^p) = L^p(\Omega,Q,\mathcal{F})$, $U^*L^p(\Omega,Q,\mathcal{F}) = \mathbb{R}\oplus L_a^p$ and the restriction of U to $\mathbb{R}\oplus L_a^p$, also denoted by U, defines a homeomorphism $U : \mathbb{R}\oplus L_a^p \to L^p(\Omega,Q,\mathcal{F})$.

We now transport by unitary equivalence the selfadjoint operator $0\oplus\mathfrak{J}$ in $\mathbb{R}\oplus L_a^2$ to a positive selfadjoint operator $J = U(0\oplus\mathfrak{J})U^*$ in $L^2(\Omega,Q,\mathcal{F})$. For X in the domain of J we have

$$JX = \sum_{i\in\mathbb{N}^*} \int_0^{\bar{T}} (\mathfrak{J}x)_t^i d\bar{W}_t^i, \tag{3.7}$$

where x is given by Lemma 3.2. For $p\geq 2$ and $s\geq 0$, the vector space $\mathbb{D}_s^p = U(\mathbb{R}\oplus L_{a,s}^p)$ is according to Lemma 3.2 a subspace of $L^p(\Omega,Q,\mathcal{F})$ and $\mathbb{D}_0^p = L^p(\Omega,Q,\mathcal{F})$. Due to the corresponding properties of $\mathfrak{J}$, for $a\geq 0$, it follows that J^a with domain $\mathbb{D}_{s+a}^p$ is a closed operator in $\mathbb{D}_s^p$. $\mathbb{D}_s^p$ becomes a Banach space for the norm defined by:

$$\|X\|_{\mathbb{D}_s^p} = (\|X\|_{L^p(\Omega,Q)}^p + \|J^sX\|_{L^p(\Omega,Q)}^p)^{1/p}. \tag{3.8}$$

We then define a decreasing family of Fréchet spaces

$$\mathsf{D}_s = \cap_{p\geq 2}\mathbb{D}_s^p, \tag{3.9}$$

$s\geq 0$ of contingent claims. Let $\mathsf{D}_\infty = \cap_{s\geq 0}\mathsf{D}_s$.

Example 3.3 *Let us consider the simple case of a deterministic volatility operator σ and a binary option with discounted pay-off X, where $X = 0$ for $\bar{p}_{\bar{T}}(T) < K$ and $X = 1$ for $\bar{p}_{\bar{T}}(T) \geq K$, with $T > 0$ and $K > 0$ given. An explicit calculus of the discounted value $E_Q(X|\mathcal{F}_t)$ and Itô's lemma give that $X = U(c,x)$, with $c = E_Q(X)$ and $x_t^i = g(K/\bar{p}_t(T+\bar{T}-t))\sigma_t^i(T+\bar{T}-t)$, where g is a continuous bounded function. If $\int_0^{\bar{T}}\sum_{i\in\mathbb{N}^*} i^{2s}\|\sigma_t^i\|_{\tilde{H}}^2 dt < \infty$, then $X\in\mathsf{D}_s$.*

Under certain conditions, D_s will turn out to be a space of allowed contingent claims in a complete market if s is sufficiently large. The space D_s is then satisfactory from the point of view of hedging contingent claims since it contains commonly (and less commonly) used derivatives, including standard and exotic bond and interest rate options. We remind that the pay-off for such options can be expressed as a function of the price (curves) $\bar{p}$. However, if $s > 0$ then D_s is not closed under multiplication:

Remark 3.4 *The space* D_s, $s > 0$, *is not closed under multiplication. In fact there exists* $X \in \mathsf{D}_\infty$ *such that for all* $s > 0$, $X^2 \notin \mathbb{D}_s^2$. *We now construct such a* X. *Let* $c^i = ((1+i)^{1/2}\log(1+i))^{-1}$ *and* $a_t = \sum_{i\in\mathbb{N}^*} c^i \bar{W}_t^i$, *which is well-defined since* $\sum_{i\in\mathbb{N}^*}(c^i)^2 < \infty$. *We set* $X = \int_0^{\bar{T}} a_t d\bar{W}_t^1$. *It follows from (3.8), (3.7) and the definition of* X *that* $\|X\|_{\mathbb{D}_s^p}^p = 2E_Q(|X|^p)$, *for* $s \geq 0$ *and* $p \geq 2$. *Let* $Y_t = \int_0^t a_s d\bar{W}_s^1$ *and* $b = \sup_{t\in\mathbb{T}} |a_t|$. *It follows using the Burkholder-Davis-Gundy (BDG) inequalities, that for some constants* C_p *and* C'_p :

$$\|X\|_{\mathbb{D}_s^p}^p \leq C_p E_Q((\int_0^{\bar{T}} (a_t)^2 dt)^{p/2}) \leq C_p \bar{T}^{p/2} E_Q(b^p) \leq C'_p (\sum_{i\in\mathbb{N}^*}(c^i)^2)^{p/2} < \infty.$$

One checks that

$$X^2 = E_Q(X^2) + 2\int_0^{\bar{T}} a_t Y_t d\bar{W}_t^1 + 2\int_0^{\bar{T}} (\bar{T}-t) a_t \sum_{i\in\mathbb{N}^*} c^i d\bar{W}_t^i.$$

The two first terms on the right hand side are in $\mathbb{D}_s^p$ *for all* $s \geq 0$ *and* $p \geq 2$. *However, if* $s > 0$ *then* $\sum_{i\in\mathbb{N}^*}(i^s c^i)^2$ *diverges, so the third term on the right hand side is not in* $\mathbb{D}_s^2$.

The fact that D_s is not closed under multiplication, is a serious draw back for the construction of optimal portfolios, such as considered in [7]. Therefore we shall introduce a decreasing family of Fréchet spaces D_s^1, $s \geq 0$, where $\mathsf{D}_s^1 \subset \mathsf{D}_s$ and D_s^1 is an algebra under multiplication. The measure ν is atomless, so the Gaussian process $\{(h, \bar{W}(h)) \,|\, h \in L^2(\mathbb{T}\times\mathbb{N}^*)\}$, where

$$\bar{W}(h) = \sum_{i\in\mathbb{N}^*} \int_0^{\bar{T}} h(s,i) d\bar{W}_s^i, \tag{3.10}$$

is well-defined (cf. [12]). The Malliavin derivative operator D, is also well-defined on smooth random variables:

$$D_{(t,i)}X = \sum_{l=1}^n f_l(\bar{W}(h_1), \ldots, \bar{W}(h_n)) h_l(t,i), \tag{3.11}$$

where $X = f(\bar{W}(h_1), \ldots, \bar{W}(h_n))$, $(t,i) \in \mathbb{T}\times\mathbb{N}^*$, $n \in \mathbb{N}^*$, $f \in C^\infty(\mathbb{R}^n)$ is polynomially bounded together with all its derivatives and $f_l(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_l}(x_1, \ldots, x_n)$. For all $1 \leq p < \infty$, the linear map in (3.11) defines a closed linear map, also denoted D, from $L^p(\Omega, Q, \mathcal{F})$ to $L^p(\Omega, Q, \mathcal{F}, L^2(\mathbb{T}\times\mathbb{N}^*))$, with dense domain $\mathbb{D}^{1,p}$ (cf. [12]). $D_t X$ denotes the ℓ^2 valued random variable defined by the canonical isomorphism $L^2(\mathbb{T}\times\mathbb{N}^*) \cong L^2(\mathbb{T}, \ell^2)$. We denote by

$\mathcal{D}_0$ the subset of random variables X in (3.11) also satisfying the restrictions that f has compact support and all the h_i are finite sequences.

For $p \geq 2$ and for $s \geq 0$, $D\mathcal{D}_0 \subset L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2}))$, so the operator $\mathfrak{J}^s D$ from $L^p(\Omega, Q, \mathcal{F})$ to $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$ is densely defined. It is also closed. In fact, let $\{X_n\}_{n\geq 1} \subset \mathcal{D}_0$ converge to X in $L^p(\Omega, Q, \mathcal{F})$ and let $\mathfrak{J}^s DX_n = y_n \to y$ in $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$. The inverse of $\mathfrak{J}^s$ in $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$ is a continuous operator. If $\mathfrak{J}^s x_n = y_n$ and $\mathfrak{J}^s x = y$, then $x \in L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2}))$, the domain of $\mathfrak{J}^s$, and $x_n \to x$ in $L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$. D is closed, so $x = DX$ and therefore $y = \mathfrak{J}^s DX$. The domain $\mathbb{D}_s^{1,p}$ of $\mathfrak{J}^s D$ becomes a real Banach space for the norm defined by

$$\|X\|_{\mathbb{D}_s^{1,p}} = (\|X\|^p_{L^p(\Omega,Q)} + \|\mathfrak{J}^s DX\|^p_{L^p(\Omega,Q,L^2(\mathbb{T},\ell^2))})^{1/p}. \tag{3.12}$$

We note that $\mathbb{D}_0^{1,p} = \mathbb{D}^{1,p}$. The decreasing family of Fréchet spaces of contingent claims D_s^1, $s \geq 0$ is defined by

$$\mathsf{D}_s^1 = \cap_{p\geq 2} \mathbb{D}_s^{1,p}. \tag{3.13}$$

One easily checks that multiplication is continuous in D_s^1 and that $\mathsf{D}_\infty^1 = \cap_{s\geq 0} \mathsf{D}_s$ is dense in D_0^1, which is dense both in $L^2(\Omega, Q, \mathcal{F})$ and in $L^2(\Omega, P, \mathcal{F})$.

Remark 3.5 *Due to its algebraic structure the space D_s^1 is suitable for solving optimal portfolio problems. However it does not include all commonly used derivative products. For example the only binary options in $\mathbb{D}^{1,2}$ are the trivial ones $X = 1$ and $X = 0$. The space D_s does not have this shortcoming (see Example 3.3). This indicates that the spaces D_s and $\mathbb{D}_s^p$ are more appropriate than the spaces D_s^1 and $\mathbb{D}_s^{1,p}$ when considering general hedging problems.*

The Clark-Ocone representation (see [12]), used in [7] (see §3.2 and Lemma A.5 of [7]) in a context of portfolio optimization, generalizes to $\mathbb{D}^{1,2}$, so in particular to D_s^1 :

Lemma 3.6 *If $X \in \mathbb{D}^{1,2}$ and x is given by the isomorphism of Lemma 3.2, then $x_t = E_Q(D_t X \mid \mathcal{F}_t)$, $t \in \mathbb{T}$.*

We omit the proof of this lemma, since it is so similar to the proof of the analog result for a one dimensional Brownian motion (see Proposition 1.3.5 of [12]). The space of contingent claims $\mathbb{D}_s^{1,p}$ is smaller than $\mathbb{D}_s^p$:

Corollary 3.7 *Let $p \geq 2$ and $s \geq 0$. Then $\mathbb{D}_s^{1,p} \subset \mathbb{D}_s^p$, $\mathsf{D}_s^1 \subset \mathsf{D}_s$ and the inclusion maps are continuous. Moreover $X \in \mathsf{D}_s^1$ if and only if $X \in \mathsf{D}_0$ and $DX \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$.*

4 Main results

In order to find a hedging portfolio $\theta \in \mathsf{P}_{sf}$ of a contingent claim X, we have according to (2.23) to solve the equation $< \theta_t\,,\, \bar{p}_t \sigma_t^i >= x_t^i$, for all $i \in \mathbb{N}^*$ or equivalently

$$(b_t(\omega))'\theta_t(\omega) = x_t(\omega), \tag{4.1}$$

a.e. $(t, \omega) \in \mathbb{T} \times \Omega$, where $b_t(\omega) = \bar{p}_t(\omega)\sigma_t(\omega) \in L(\ell^2, H)$ is the martingale operator at (t, ω) and x is given by the martingale decomposition of X in Lemma 3.2. Let $(b_t(\omega))^*$ be the adjoint of $b_t(\omega)$ with respect to the scalar product in H. Using that the operators $b_t(\omega)^*$ and $(b_t(\omega)^* b_t(\omega))^{1/2}$ have the same range a.e. $(t, \omega) \in \mathbb{T} \times \Omega$, (cf. Lemma A.1) we prove, that derivatives in L^p, $p \geq 1$ and even derivatives in D_0 are not always attainable:

Theorem 4.1 *If condition I is satisfied, then there exists $X \in \mathsf{D}_0$ such that $\bar{V}_{\bar{T}}(\theta) \neq X$ for all $\theta \in \mathsf{P}_{sf}$.*

For an example and generalizations, see Remark 4.6. The bond market is approximatively complete in the following sense:

Theorem 4.2 *Let condition I be satisfied. D_0 has a dense subspace of attainable contingent claims if and only if the operator $\sigma_t(\omega)$ has a trivial kernel a.e. $(t, \omega) \in \mathbb{T} \times \Omega$.*

To introduce complete markets, we shall impose a supplementary condition on the volatility operator. We now give a motivation of this condition. The operator $B_t(\omega) = l_t \sigma_t(\omega)$, where $l_t = \mathcal{L}_t p_0$, is a.e. (t, ω) a Hilbert-Schmidt operator from ℓ^2 to H, when Condition I is satisfied. Let

$$A_t(\omega) = (B_t(\omega))^* B_t(\omega) \tag{4.2}$$

where $(B_t(\omega))^*$ is the adjoint of $B_t(\omega)$ with respect to the scalar product in H. $A_t(\omega)$ is a positive self-adjoint trace-class operator in ℓ^2 a.e. (t, ω), when Condition I is satisfied. In particular the operator $A_t(\omega)$ in ℓ^2 is compact a.e. (t, ω), so it can not have a bounded inverse. However it can have an inverse defined on $\ell^{s,2}$, for some $s > 0$. This simple observation leads us to replace the non-degeneracy condition, which gives complete markets in the case of a finite number of random sources (see [7] formula (3.8) and Remark 5.3), by the following:

Condition II *There exists $s > 0$ and $k \in \mathsf{D}_0$, such that for all $x \in \ell^2$:*

$$\|x\|_{\ell^2} \leq k(\omega) \|(A_t(\omega))^{1/2} x\|_{\ell^{s,2}} \text{ a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \tag{4.3}$$

As we will see, if Condition II is satisfied and $X \in \mathsf{D}_s$, then a.e. (t, ω) the equation

$$B_t^*(\omega)\eta_t(\omega) = x_t(\omega) \tag{4.4}$$

has a solution in H given by

$$\eta_t(\omega) = S_t(\omega)(A_t(\omega))^{-1/2}x_t(\omega), \tag{4.5}$$

where $S_t(\omega)$, the closure of $B_t(\omega)(A_t(\omega))^{-1/2}$, is an isometric operator from ℓ^2 to H. Let $\mathcal{S} \in L(\tilde{H}, \tilde{H}')$ be defined by

$$(f, g)_{\tilde{H}} =< \mathcal{S}f \, , \, g >, \tag{4.6}$$

for $f, g \in \tilde{H}$. The portfolio θ^1, given by

$$\theta_t^1 = (l_t/\bar{p}_t)\mathcal{S}\eta_t \tag{4.7}$$

then satisfies equation (4.1) and gives the risky part of a self-financing portfolio $\theta = \theta^0 + \theta^1 \in \mathsf{P}_{sf}$. Here $\theta^0 \in \mathsf{P}$ is a portfolio of zero-coupon bonds with time to maturity 0 and

$$\theta_t^0 = a_t\delta_0, \tag{4.8}$$

where δ_0 is the Dirac measure with support at 0 and a is the unique real valued process such that θ is self-financing. Heuristically, this leads to the completeness of the market, when the allowed contingent claims are given by D_s and the conditions I and II are satisfied:

Theorem 4.3 *If Condition I and Condition II are satisfied and if $X \in \mathsf{D}_s$, where $s > 0$ is given by Condition II, then there exists $\theta \in \mathsf{P}_{sf}$ such that $\bar{V}_{\bar{T}}(\theta) = X$. Moreover, one such portfolio is $\theta = \theta^0 + \theta^1$, where $\theta^0, \theta^1 \in \mathsf{P} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$ are given by formulas (4.7) and (4.8). The linear mappings $\mathsf{D}_s \ni X \mapsto \theta^i \in \mathsf{P} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$, $i = 0, 1$ are continuous.*

This theorem has a converse:

Theorem 4.4 *Let Condition I be satisfied and assume that there exist $s \geq 0$ and $k \in \mathsf{D}_0$, such that for all $x \in \ell^{s,2}$, $\|(A_t(\omega))^{1/2}\mathfrak{j}^s x\|_{\ell^2} \leq k(\omega)\|x\|_{\ell^2}$, a.e. $(t, \omega) \in \mathbb{T} \times \Omega$. Then $\bar{V}_{\bar{T}}(\theta) \in \mathsf{D}_s$, for all $\theta \in \mathsf{P}_{sf} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$.*

We shall apply these results to the optimal bond portfolio problem considered in [7], which we now introduce. The set of all admissible self-financing portfolios with initial wealth x is

$$\mathcal{C}(x) = \{\theta \in \mathsf{P}_{sf} \mid \bar{V}_0(\theta) = x\}.$$

The optimization problem is, for a given initial wealth K_0, to find a solution $\hat{\theta} \in \mathcal{C}(K_0)$ of

$$E(u(\bar{V}_{\bar{T}}(\hat{\theta}))) = \sup_{\theta \in \mathcal{C}(K_0)} E(u(\bar{V}_{\bar{T}}(\theta))) \tag{4.9}$$

where the utility function u satisfies the following Inada-type condition:

Condition III
a) $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly concave, upper semi-continuous and finite on an interval $]\underline{x}, \infty[$, with $\underline{x} \leq 0$ (the value $\underline{x} = -\infty$ is allowed).

b) u is C^2 on $]\underline{x}, \infty[$ and $u'(x) \to \infty$ when $x \to \underline{x}$ in $]\underline{x}, \infty[$.

c) there exist some $q > 0$ and $C > 0$ such that

$$\liminf_{x \downarrow \underline{x}} (1 + |x|)^{-q} u'(x) > 0 \tag{4.10}$$

and such that, if $u' > 0$ on $]\underline{x}, \infty[$ then

$$\limsup_{x \to \infty} x^q u'(x) < \infty \quad \text{and} \quad |x\varphi'(x)| \leq C(x^q + x^{-q}) \text{ for all } x > 0 \tag{4.11}$$

and if u' takes the value zero then

$$\limsup_{x \to \infty} x^{-q} u'(x) < 0 \quad \text{and} \quad |x\varphi'(x)| \leq C(1 + |x|)^q \text{ for all } x \in \mathbb{R}, \tag{4.12}$$

where φ is the inverse of u' restricted to $]\underline{x}, \infty[$.

Theorem 4.5 *Let Condition I, Condition II and Condition III be satisfied and let $\ln(\xi_{\bar{T}}) \in \mathsf{D}^1_s$, where $s > 0$ is given by Condition II. If $K_0 \in]\underline{x}, \infty[$, then problem (4.9) has a solution $\hat{\theta}$.*

We end this section with the following remarks:

Remark 4.6 *One can consider stronger formulations of Theorem 4.1. For example, whether or not one can choose the non-attainable claim X bounded and smooth is an open question in general. For constant deterministic volatility operators the answer is yes. In fact, the operator $A_t(\omega) = A$ (see (4.2)) in ℓ^2 is then constant (t, ω) and since $A^{1/2}$ is compact, we can choose $e \in \ell^2$, such that $\|e\|_{\ell^2} = 1$ and $e \notin \mathcal{R}(A^{1/2})$. Let $g \in C^\infty(\mathbb{R})$ be rapidly decreasing together with all its derivatives, $g(x) > 0$ for all $x \in \mathbb{R}$, $f(x) = \int_{y \leq x} g(y)dy$, $Y = \sum_{n \geq 1} e^n \bar{W}^n_{\bar{T}}$ and $X = f(Y)$. Then, by Lemma 3.6, $X = E_Q(X) + \int_0^{\bar{T}} \sum_{n \geq 1} x^n_t d\bar{W}^n_t$, where $x_t = E_Q(g(Y) \,|\, \mathcal{F}_t)e$. Since $e \notin \mathcal{R}(A^{1/2})$ and $E_Q(g(Y) \,|\, \mathcal{F}_t) > 0$ for all t, it follows that X is not attainable.*

Remark 4.7 *Our choice $\ell^{s,2}$, $s \geq 0$ of weighted ℓ^2-spaces, leading to the results of this section, can be generalized to other weighted ℓ^2-spaces. If q_s are the corresponding norms, then the crucial property which shall be satisfied is Condition II with the $\ell^{s,2}$-norms replaced by q_s, which can depend on (t, ω).*

Remark 4.8 *Conditions can be given directly on Γ, which guarantees that $\ln(\xi_{\bar{T}})$ satisfies the hypothesis of Theorem 4.5. One possibility is: If $s \geq 0$ and if for all $n \in \{0, 1, 2\}$ and $p \geq 2$, $D^n\Gamma \in L^p(\Omega, Q, \otimes^{n+1} L^2(\mathbb{T}, \ell^{s,2}))$, then $\ln(\xi_{\bar{T}}) \in \mathsf{D}_s^1$.*

5 Proofs

Proof of Lemma 3.1 Let $p \geq 2$. It follows from Schwarz inequality that $\|X\|_{L^p(P,F)} = (E_Q(\xi_{\bar{T}}^{-1}\|X\|_F^p))^{1/p} \leq (E_Q(\xi_{\bar{T}}^{-2}))^{1/2p}\|X\|_{L^{2p}(Q,F)}$. Similarly, $\|X\|_{L^p(Q,F)} \leq (E(\xi_{\bar{T}}^2))^{1/2p}\|X\|_{L^{2p}(P,F)}$. According to Theorem 2.2, $E_Q(\xi_{\bar{T}}^{-2}) = E(\xi_{\bar{T}}^{-1}) < \infty$ and $E(\xi_{\bar{T}}^2) < \infty$. **QED**

Proof of Corollary 3.7 Let p and s be as in the corollary and let $X \in \mathbb{D}_s^{1,p}$. Obviously $X \in \mathbb{D}^{1,2}$, so we can apply Lemma 3.6 giving $X = U(c, x)$, where $x_t = E_Q(D_tX \mid \mathcal{F}_t)$, $t \in \mathbb{T}$. We obtain $\|\mathfrak{J}^s x\|_{L^p(\Omega,Q,L^2(\mathbb{T},\ell^2))} \leq \|\mathfrak{J}^s D_t X\|_{L^p(\Omega,Q,L^2(\mathbb{T},\ell^2))} \leq \|X\|_{\mathbb{D}_s^{1,p}}$. Then $x \in L_{a,s}^p$, so by definition $X \in \mathbb{D}_s^p$, proving tat $\mathbb{D}_s^{1,p} \subset \mathbb{D}_s^p$. The inclusion map is continuous since, by the last inequality and Lemma 3.2, for some constant C_p we have $\|J^s X\|_{L^p(\Omega,Q)} \leq C_p\|\mathfrak{J}^s x\|_{L^p(\Omega,Q,L^2(\mathbb{T},\ell^2))} \leq C_p\|X\|_{\mathbb{D}_s^{1,p}}$.

The continuous inclusion of D_s^1 into D_s is now a direct consequence of the continuous inclusion of $\mathbb{D}_s^{1,p}$ into $\mathbb{D}_s^p$, for all $p \geq 2$.

According to formula (3.12), $X \in \mathsf{D}_s^1$ if and only if $X \in L^p(\Omega, Q)$ and $\mathfrak{J}^s X \in L^p(\Omega, Q, L^2(\mathbb{T}, \ell^2))$, for all $p \geq 2$. Then by Lemma 3.1, $X \in \mathsf{D}_s^1$ if and only if $X \in \mathsf{D}_0$ and $DX \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$. **QED**

Proof of Theorem 4.1 Using the definition (4.6) of $\mathcal{S}$, one readily verifies that the operator $(b_t(\omega))'$ in equation (4.1) satisfies

$$(b_t(\omega))'\mathcal{S} = (b_t(\omega))^*, \tag{5.1}$$

a.e. (t, ω). Since $\mathcal{S}$ defines a homeomorphism of H' onto H, equation (4.1) is equivalent to find a H-valued process y satisfying $\mathcal{S}y \in \mathsf{P}_{sf}$ and $(b_t(\omega))^* y_t(\omega) = x_t(\omega)$, a.e. $(t, \omega) \in \mathbb{T} \times \Omega$. A necessary condition for the existence of a solution of equation (4.1) is then according to Lemma A.1:

$$x_t(\omega) \in \mathcal{R}(((b_t(\omega))^* b_t(\omega))^{1/2}), \text{ a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \tag{5.2}$$

Let $K_t(\omega) = ((b_t(\omega))^* b_t(\omega))^{1/2}$, let $\|K_t(\omega)\|_{H-S}$ be its Hilbert-Schmidt norm and let $\{u_n\}_{n\geq 1}$ be the standard orthonormal basis in ℓ^2. Then $\|K_t(\omega)\|_{H-S}^2 =$

$\sum_{i\geq 1}\|b_t(\omega)u_i\|^2_H \leq C\sup_s \|\bar{p}_s(\omega)\|^2_H \sum_i \|\sigma^i_t(\omega)\|^2_{\tilde{H}}$, which is integrable according to Hölder's inequality, inequality (2.19) of Condition I and Theorem 2.1. Therefore $\|K_t(\omega)\|^2_{H-S}$ is finite a.e. (t,ω). We can now apply Lemma A.4 and choose $x_t(\omega) = g_1(K_t(\omega))$, a.e. $(t,\omega)\in\mathbb{T}\times\Omega$. Then $x_t(\omega)\notin\mathcal{R}(K_t(\omega))$ a.e. $(t,\omega)\in\mathbb{T}\times\Omega$. Since g_1 is Borel measurable on the space of selfadjoint Hilbert-Schmidt operators, with the operator norm topology, it follows that x is progressive.

Let $X = U(0,x)$, where U is as in Lemma 3.2. It follows from Lemma A.4, Lemma 3.1 and Lemma 3.2 that $x\in\mathsf{D}_0$. Since condition (5.2) is not satisfied, it follows that equation (4.1) does not have a solution for this x. **QED**

In the proof of the next two theorems we shall use the following

Lemma 5.1 *If Condition I is satisfied and if $\theta^1\in\mathsf{D}_0(L^2(\mathbb{T},H'))$ and $x\in\mathsf{D}_0(L^2(\mathbb{T},\ell^{s,2}))$ are progressively measurable processes satisfying formula (4.1), then $\theta^1\in\mathsf{P}$. If, moreover $c\in\mathbb{R}$,*

$$Y_t = c + \sum_{i\in\mathbb{N}^*}\int_0^t x^i_s d\bar{W}^i_s, \tag{5.3}$$

$$a_t = (\bar{p}_t(0))^{-1}(Y_t - <\theta^1_t\,,\,\bar{p}_t>) \text{ and } \theta^0_t = a_t\delta_0, \tag{5.4}$$

for $t\in\mathbb{T}$ and $Z=\sup_{t\in\mathbb{T}}|Y_t|$, then Y is a Q-martingale, $Z\in\mathsf{D}_0$, $a\in\mathsf{D}_0(L^2(\mathbb{T}))$, $\theta^0\in\mathsf{P}\cap\mathsf{D}_0(L^2(\mathbb{T},H'))$ and $\theta=\theta^0+\theta^1\in\mathsf{P}_{sf}\cap\mathsf{D}_0(L^2(\mathbb{T},H'))$. The linear map $(\theta^1,c,x)\mapsto a\in\mathsf{D}_0(L^2(\mathbb{T}))$ is continuous on the subspace of progressively measurable processes in $\mathsf{D}_0(L^2(\mathbb{T},H'))\times\mathbb{R}\times\mathsf{D}_0(L^2(\mathbb{T},\ell^{s,2}))$ satisfying (4.1).

Proof: Since θ^1 satisfies $b'\theta^1\in\mathsf{D}_0(L^2(\mathbb{T},\ell^{s,2}))$ by construction, it follows from the definition of P, Condition I, $\theta^1\in\mathsf{D}_0(L^2(\mathbb{T},H'))$, and Hölder's inequality that $\theta^1\in\mathsf{P}$.

Obviously Y is a Q-martingale. By Doob's L^p-inequality (cf. [14]) we have $\|Z\|_{L^p(Q)}\leq c_p\sup_{t\in\mathbb{T}}\|Y_t\|_{L^p(Q)}$, for $p\geq 2$. We then obtain $\|Z\|_{L^p(Q)}\leq c_p\|X\|_{L^p(Q)}$, since $|Y|^p$ is a Q-submartingale. Lemma 3.1 now gives that

$$Z\in\mathsf{D}_0. \tag{5.5}$$

Schwarz inequality and the definition of Z give

$$(\int_0^{\bar{T}}|a_t|^2dt)^{1/2}\leq(\sup_{t'\in\mathbb{T}}|\bar{p}_{t'}(0)|^{-1})(Z\bar{T}+(\sup_{t'\in\mathbb{T}}\|\bar{p}_{t'}\|_H)(\int_0^{\bar{T}}\|\theta^1_t\|^2_{H'}dt)^{1/2}).$$

Formula (5.5), $\theta^1 \in \mathsf{D}_0(L^2(\mathbb{T}, H'))$ and Hölder's inequality then give

$$a \in \mathsf{D}_0(L^2(\mathbb{T})) \tag{5.6}$$

and the announced continuity property of a. Since $\|\delta_t\|_{H'} = C < \infty$, for $t \in \mathbb{T}$ and C independent of t, it follows from formulas (2.11) and (2.20) that $\|\theta^1\|_{\mathsf{P}} = C\|a\|_{L^2(\Omega,P,L^2(\mathbb{T}))}$. Formula (5.6) now shows that $\theta^0 \in \mathsf{P} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$.

By the definition (5.4) of a, it follows that $\bar{V}_t(\theta) = Y_t$. θ is then self-financing according to formulas (2.23) and (5.3) with $\bar{V}_0(\theta) = c$. **QED**

The following notations will be used in the proof of Theorem 4.2: $d\mu = dtdQ$. F is the closed subspace of progressively measurable processes in $L^2(\Omega, Q, L^2(\mathbb{T}, H))$. For $p > 1$, the operator $b_{(p)}$ from L^p_a to F is defined by its domain $\mathcal{D}(b_{(p)}) = \{x \in L^p_a \mid \int_{\mathbb{T}\times\Omega} \|b_t(\omega)x_t(\omega)\|^2_H d\mu < \infty\}$ and

$$(b_{(p)}x)_t(\omega) = b_t(\omega)x_t(\omega), \tag{5.7}$$

where b is as in (4.1). The operator $b_{(p)}$ is densely defined and closed. We note that $b_{(p)}$, $p > 1$ is a maximal operator in the sense that it does not have a nontrivial extension, satisfying (5.7). The adjoint $(b_{(p)})^*$ is given by $((b_{(p)})^*y)_t(\omega) = (b_t(\omega))^*y_t(\omega)$, $1/p + 1/q = 1$ and $\mathcal{D}((b_{(p)})^*) = \{y \in F \mid E_Q((\int_{\mathbb{T}} \|(b_t(\omega))^*y_t(\omega)\|^2_H)^{q/2}) < \infty\}$. Given a selfadjoint operator A, we denote by e_A be the resolution of the identity associated with A.

Proof of Theorem 4.2 Let $\mathcal{U} = \{(t,\omega) \in \mathbb{T} \times \Omega \mid \mathcal{K}(\sigma_t(\omega)) \neq \{0\}\}$.

1) Let $\mu(\mathcal{U}) = 0$. The set $\mathcal{D}_1 = \cap_{p>1}\mathcal{D}^{(p)}$, where $\mathcal{D}^{(p)} = \{y \in F \mid y \in \mathsf{D}_0(L^2(\mathbb{T}, H)), b^*_{(p)}y \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^2))\}$ is dense in F. In fact its enough to consider progressive $y \in L^\infty(\Omega \times \mathbb{T}, H))$.

For $y \in \mathcal{D}_1$, we define $\theta^1 = \mathcal{S}y$. The relation (5.1) gives $(b_t(\omega))'\theta^1_t(\omega) = (b^*_{(p)}y)_t(\omega)$. Then, according to the definition of $\mathcal{D}_1$, the hypotheses of Lemma 5.1 are satisfied with $x_t(\omega) = (b_t(\omega))'\theta^1_t(\omega)$. θ^0 is defined by (5.4). Lemma 5.1 gives $\theta^0, \theta^1 \in \mathsf{P}$ and $\theta = \theta^0 + \theta^1 \in \mathsf{P}_{sf}$. Let $\mathcal{D}_2 \subset \mathsf{P}_{sf}$ be the set of all such θ, for $y \in \mathcal{D}_1$. By construction $\bar{V}_{\bar{T}}(\theta) = U(\bar{V}_0(\theta), x)$ so $\theta \mapsto (\bar{V}_0(\theta), x)$ defines a mapping of $\mathcal{D}_2$ onto $\mathbb{R} \oplus b^*_{(p)}\mathcal{D}_1$. $\bar{V}_{\bar{T}}(\theta) \in \mathsf{D}_0$ according to Lemma 3.2, since $x \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$.

For the moment suppose that, for every $p > 1$, $(b_{(p)})^*\mathcal{D}_1$ is dense in L^q_a, the dual of L^p_a, where $p^{-1} + q^{-1} = 1$. Since the set $\mathcal{D}_3 = (b_{(p)})^*\mathcal{D}_1$ is independent of p, it follows that $\mathcal{D}_3$ is dense in $\mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$. By Lemma 3.2 it then follows that the set of attainable claims $U(\mathbb{R} \oplus \mathcal{D}_3)$ is dense in D_0.

It remains to prove that $(b_{(p)})^*\mathcal{D}_1$ is dense in L^q_a. Let $c_{(p)}$ be the restriction of $(b_{(p)})^*$ to $\mathcal{D}_1$ and let $\tilde{b}_{(p)} = (c_{(p)})^*$. Then $\tilde{b}_{(p)}$ is an extension of $b_{(p)} = (b_{(p)})^{**}$

and it satisfy (5.7) with $\tilde{b}_{(p)}$ instead of $b_{(p)}$, which shows that $b_{(p)} = \tilde{b}_{(p)}$. Therefore $\mathcal{K}(b) = \mathcal{R}(c)^{\perp}$. Since $\mathcal{K}(b)$ is trivial, $\mathcal{R}(c)$ is dense in L^q_a.

2) Let $\mu(\mathcal{U}) > 0$. We proceed as in the proof of Theorem 4.1, introduce the selfadjoint operator $K_t(\omega) = ((b_t(\omega))^* b_t(\omega))^{1/2}$ in ℓ^2, apply Lemma A.4 and choose $x_t(\omega) = g_0(K_t(\omega))$. Then $0 \neq x \in L^p_a$ for all $p > 1$ and $X = U(0,x) \in \mathsf{D}_0$. K with domain $\mathcal{D}(b_{(2)})$ is selfadjoint in L^2_a. One readily verifies that x is orthogonal to $\mathcal{R}(K)$ in L^2_a. This proves that $X = U(0,x)$ is orthogonal to the image of $\mathcal{R}(K)$ under $U(0,\cdot)$, so L^2_a does not have a dense subspace of attainable elements. This is then also the case of D_0. **QED**

Proof of Theorem 4.3 Let the conditions of the theorem be satisfied. Then $X = U(c,x)$ for some $c \in \mathbb{R}$ and $x \in \cap_{p \geq 2} L^p_{a,s}$, according to the construction of D_s. We choose x progressively measurable by changing it on a set of zero measure. We observe that $\cap_{p\geq 2} L^p_{a,s} \subset \cap_{p \geq 2} L^p(\Omega, Q, L^2(\mathbb{T}, \ell^{s,2})) \subset \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$, where the last relation follows from Lemma 3.1. This shows that

$$x \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2})), \tag{5.8}$$

where x is progressively measurable.

Let $(t,\omega) \in \mathbb{T} \times \Omega$ be such that $B_t(\omega) \in L(\ell^2, H)$. Inequality (4.3) implies that $(A_t(\omega))^{1/2} \in L(\ell^2)$ has a trivial kernel. Lemma A.1 then firstly shows that $B_t(\omega)$ also has a trivial kernel and secondly shows that $(A_t(\omega))^{-1/2}$ is densely defined and that $S_t(\omega)$ in formula (4.5) is isometric from ℓ^2 to H. According to inequality (4.3), if $z \in \ell^{s,2}$ then z is in the domain of $(A_t(\omega))^{-1/2}$ and $\|(A_t(\omega))^{-1/2} z\|_{\ell^2} \leq k_t(\omega) \|z\|_{\ell^{s,2}}$. By equation (4.5) we get $\|\eta_t(\omega)\|_{\ell^2} \leq k_t(\omega)\|x_t(\omega)\|_{\ell^{s,2}}$. Since this is true a.e. (t,ω) it follows by integration with respect to P, from Condition II and Hölder's inequality, that $\eta \in \mathsf{D}_0(L^2(\mathbb{T}, H))$. η is progressively measurable since this is the case of x and σ. In fact, if y is a progressively measurable ℓ^2 valued process then this is also the case for $A^{-1/2}\mathfrak{j}^{-s} y$, according to Lemma A.3. With $y = \mathfrak{j}^{-s} x$, it follows that $A^{-1/2} x$ is progressively measurable and then from Lemma A.3 that η given by (4.5) is progressively measurable. Let θ^1 be given by equation (4.7). Using now that $\mathcal{S}$ is unitary, that $\|\theta^1_t\|_{H'} \leq \|l_t/\bar{p}_t\|_{\tilde{H}} \| \mathcal{S}\eta_t\|_{H'}$ and using Theorem 2.1 and Hölder's inequality, it follows that

$$\theta^1 \in \mathsf{D}_0(L^2(\mathbb{T}, H')), \tag{5.9}$$

where θ^1 is progressively measurable. Since θ^1 satisfies equation (4.1) by construction and formulas (5.8) and (5.9) hold, the hypotheses of Lemma 5.1 are satisfied, so $\theta^1 \in \mathsf{P}$. It also follows that the mapping $\mathsf{D}_s \ni X \mapsto \theta^1 \in \mathsf{P} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$ is continuous.

We define a as in formula (5.4). Lemma 5.1 then gives that $\theta^0 \in \mathsf{P} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$, that $\theta \in \mathsf{P}_{sf}$ and that θ^0 has the announced continuity prop-

erty. **QED**

Proof of Theorem 4.4 Let the hypotheses of the theorem be satisfied and let $\theta \in \mathsf{P}_{sf} \cap \mathsf{D}_0(L^2(\mathbb{T}, H'))$. According to Theorem 2.2, $\bar{V}_{\bar{T}}(\theta) \in L^2(\Omega, Q, \mathcal{F})$. The self-financing condition (2.23) and Lemma 3.2, show that $\bar{V}_{\bar{T}}(\theta) = U(c, x)$, where $c \in \mathbb{R}$ and $x \in L^2_a$ is given by formula (4.1). Obviously $c \in \mathsf{D}_s$, so we only have to prove that $U(0, x) \in \mathsf{D}_s$. By the construction of D_s it is enough to prove that $x \in L^p_{a,s}$, for all $p \geq 2$, which is equivalent to that x is progressively measurable and $\mathfrak{J}^s x \in L^p(Q, L^2(\mathbb{T}, \ell^2))$, for all $p \geq 2$, where $\mathfrak{J}$ is given by formula (3.5). As x is progressively measurable, Lemma 3.1 shows that it is sufficient to check that $\mathfrak{J}^s x \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^2))$.

For the moment let us suppose that a.e. (t, ω),

$$\mathfrak{j}^s (B_t(\omega))' \in L(\ell^2, H') \quad \text{and} \quad \|\mathfrak{j}^s (B_t(\omega))'\| \leq k(\omega), \tag{5.10}$$

where the norm is the operator norm. Since $(b_t(\omega))' = (B_t(\omega))' \bar{q}_t(\omega)$, where $\bar{q}_t(\omega) = \bar{p}_t(\omega)/l_t$, it follows from (5.10) that $\|\mathfrak{j}^s x_t(\omega)\|_{\ell^2} = \|\mathfrak{j}^s (B_t(\omega))' \bar{q}_t(\omega) \theta_t(\omega)\|_{\ell^2}$ $\leq k(\omega) \|\bar{q}_t(\omega) \theta_t(\omega)\|_{H'}$. Using that $\|Gf\|_{H'} \leq C \|G\|_{\tilde{H}} \|f\|_{H'}$, where C only depends on $\tilde{s}$, we obtain

$$\|\mathfrak{j}^s x_t(\omega)\|_{\ell^2} \leq k(\omega) \|\bar{q}_t(\omega)\|_{\tilde{H}} \|\theta_t(\omega)\|_{H'}.$$

Hölder's inequality, with $1/p = 1/p_1 + 1/p_2 + 1/p_3$, $p < p_1, p_2, p_3 < \infty$, gives

$$\|\mathfrak{J}^s x\|_{L^p(P, L^2(\mathbb{T}, \ell^2))} \leq C \|k\|_{L^{p_1}(P)} \|\bar{q}\|_{L^{p_2}(P, L^\infty(\mathbb{T}, \tilde{H}))} \|\theta\|_{L^{p_3}(P, L^2(\mathbb{T}, H'))}. \tag{5.11}$$

Since by hypothesis $k \in \mathsf{D}_0$ and $\theta \in \mathsf{D}_0(L^2(\mathbb{T}, H'))$, the norms of k and θ on the right hand side of (5.11) are finite. The norm of $\bar{q}$ is also finite, according to Theorem 2.1, so $\mathfrak{J}^s x \in L^p(P, L^2(\mathbb{T}, \ell^2))$, for all $p \geq 2$. This proves that $\mathfrak{J}^s x \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^2))$.

It remains to prove (5.10). If $x \in \ell^{s,2}$, then it follows from the definition of the process A and the hypothesis of the theorem that $\|B_t(\omega) \mathfrak{j}^s x\|^2_H = (\mathfrak{j}^s x, (B_t(\omega))^* B_t(\omega) \mathfrak{j}^s x)_H = \|(A_t(\omega))^{1/2} \mathfrak{j}^s x\|^2_{\ell^2} \leq (k(\omega))^2 \|x\|^2_{\ell^2}$. $B_t(\omega) \mathfrak{j}^s$ from ℓ^2 to H is then closeable and its closure $K_t(\omega) \in L(\ell^2, H)$ has norm bounded by $k(\omega)$. We have $(K_t(\omega))^* = \mathfrak{j}^s (B_t(\omega))^*$, since $B_t(\omega) \in L(\ell^2, H)$. This shows that $\mathfrak{j}^s (B_t(\omega))^* \in L(\ell^2, H)$ has norm bounded by $k(\omega)$. The relation $(B_t(\omega))' = (B_t(\omega))^* \mathcal{S}^{-1}$, where $\mathcal{S}$ is the isomorphism defined in (4.6), now gives (5.10). **QED**

Proof of Theorem 4.5 We only consider the case of $u' > 0$, since the case of $u'(x) = 0$ for some x is so similar. Let the hypotheses of the theorem be satisfied. According to Corollary 3.4 of reference [7], the portfolio $\hat{\theta}$ is a solution of equation (4.9), if $\bar{V}_{\bar{T}}(\hat{\theta}) = \hat{X}$, where $\hat{X} = \varphi(\lambda \xi_{\bar{T}})$ for a certain $\lambda > 0$. φ is C^1 and $\ln(\xi_{\bar{T}}) \in \mathsf{D}^1_s$, so $\mathfrak{j}^s D_t \hat{X} = \lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}}) \mathfrak{j}^s D_t \ln(\xi_{\bar{T}})$. This

gives $\|\mathfrak{J}^s D\hat{X}\|_{L^2(\mathbb{T},\ell^2)} = |\lambda\xi_{\bar{T}}\varphi'(\lambda\xi_{\bar{T}})|\,\|\mathfrak{J}^s D\ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T},\ell^2)}$. Inequality (4.11) gives $\|\mathfrak{J}^s D\hat{X}\|_{L^2(\mathbb{T},\ell^2)} \leq C((\lambda\xi_{\bar{T}})^p + (\lambda\xi_{\bar{T}})^{-p})\|\mathfrak{J}^s D\ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T},\ell^2)}$. Theorem 2.2 shows that $(\lambda\xi_{\bar{T}})^p + (\lambda\xi_{\bar{T}})^{-p} \in L^q(\Omega, P)$, for all $q \geq 1$. By hypothesis $\|\mathfrak{J}^s D\ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T},\ell^2)} \in \mathsf{D}_0$, so Hölder's inequality now gives that $\|\mathfrak{J}^s D\hat{X}\|_{L^2(\mathbb{T},\ell^2)} \in \mathsf{D}_0$, i.e. $D\hat{X} \in \mathsf{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$. By Theorem 3.3 of reference [7], $\hat{X} \in \mathsf{D}_0$. Corollary 3.7 then gives that $\hat{X} \in \mathsf{D}^1_s$. We can now apply Theorem 4.3, which proves the existence of $\hat{\theta}$. **QED**

A Auxiliary results

Proof of Lemma 3.2 We first prove that the mapping $(c, x) \mapsto X = U(c, x)$ is unitary on $\mathbb{R} \oplus L^2_a$ to $L^2(\Omega, Q, \mathcal{F})$.

The operator U is isometric, so its range is a closed subspace of $L^2(\Omega, Q, \mathcal{F})$. In fact (cf. Proposition 4.13 of [5]), $\|U(c, x)\|^2_{L^2(Q)} = c^2 + \|x\|^2_{L^2_a}$. It is sufficient to prove that U has dense range. Let $h \in L^2(\mathbb{T}, \ell^2)$ and let

$$\mathcal{E}_t(h) = \exp\left(-\frac{1}{2}\int_0^t \sum_{i\in\mathbb{N}^*}(h(s,i))^2 ds + \int_0^t \sum_{i\in\mathbb{N}^*} h(s,i)d\bar{W}^i_s)\right).$$

$\mathcal{E}_{\bar{T}}(h)$ is in the range of U, since $\mathcal{E}(h)h \in L^2_a$ and by Itô's lemma (Theorem 4.17 of [5]):

$$\mathcal{E}_{\bar{T}}(h) = 1 + \int_0^{\bar{T}} \sum_{i\in\mathbb{N}^*} \mathcal{E}_s(h)h(s,i)d\bar{W}^i_s.$$

We have $L^2(\mathbb{T}, \ell^2) \cong L^2(\mathbb{T} \times \mathbb{N}^*)$ and the measure ν is atomless. The linear span of $\{\mathcal{E}_{\bar{T}}(h) \,|\, h \in L^2(\mathbb{T}, \ell^2)\}$ is then dense in $L^2(\Omega, Q, \mathcal{F})$ (cf. Lemma 1.1.2 of [12]), which proves that U is a unitary operator.

To prove the second part of the lemma we fix $p \geq 2$. For $(c, x) \in \mathbb{R} \oplus L^2_a$ let $X = U(c, x)$, for $0 \leq t \leq \bar{T}$ let

$$Y_t = \sum_{i\in\mathbb{N}^*} \int_0^t x^i_s d\bar{W}^i_t$$

and let $Z = \sup_{0\leq t\leq \bar{T}} |Y_t|$. In the sequel of this proof $C, C_1, C_2, \ldots$ are positive constants independent of X and (c, x). Applying the BDG inequalities we obtain

$$\|X\|_{L^p(Q)} \leq |c| + \|Z\|_{L^p(Q)} \leq |c| + C\|x\|_{L^p_a}.$$

This shows that

$$U(\mathbb{R} \oplus L^p_a) \subset L^p(\Omega, Q, \mathcal{F}). \tag{A.1}$$

Given $X \in L^p(\Omega, Q, \mathcal{F})$, then $X \in L^2$. By the first part of the lemma it follows that $(c, x) = U^* X \in \mathbb{R} \oplus L^2_a$. Since U^* is continuous, $|c| \leq C_1 \|X\|_{L^2(Q)} \leq C_1 \|X\|_{L^p(Q)}$. The BDG inequalities give $\|x\|_{L^p_a} \leq C_2 \|Z\|_{L^p(Q)}$. Applying Doob's L^p inequalities and using that $|Y|^p$ is a submartingale, we obtain that

$$\|x\|_{L^p_a} \leq C_3 \sup_{0 \leq t \leq \bar{T}} \|Y_t\|_{L^p(Q)} \leq C_3 \|X\|_{L^p(Q)}.$$

This proves that $U^* L^p(\Omega, Q, \mathcal{F}) \subset \mathbb{R} \oplus L^p_a$. Since U is unitary it follows that $L^p(\Omega, Q, \mathcal{F}) \subset U(\mathbb{R} \oplus L^p_a)$, which together with (A.1) proves that $U(\mathbb{R} \oplus L^p_a) = L^p(\Omega, Q, \mathcal{F})$. This gives by unitarity $U^* L^p(\Omega, Q, \mathcal{F}) = \mathbb{R} \oplus L^p_a$.

Finally the restriction $B \in L(\mathbb{R} \oplus L^p_a, L^p(\Omega, Q, \mathcal{F}))$ of U is a homeomorphism since B^{-1} is the restriction of U^* to $L^p(\Omega, Q, \mathcal{F})$. **QED**

In the sequel E, E_1 and E_2 are separable Hilbert spaces. The next lemma collects some well-known results on polar decomposition, cf. Ch VI, §7 of [11]. We recall that, if K is a densely defined closed operator from E_1 to E_2 with adjoint K^*, then according to von Neumann's theorem, K^*K is a positive self-adjoint operator in E_1. Its positive square-root is then well-defined.

Lemma A.1 *Let E_1 and E_2 be Hilbert spaces and let K be a densely defined closed operator from E_1 to E_2. The following statements are true: i) $\mathcal{R}(K^*) = \mathcal{R}((K^*K)^{1/2})$ and $\mathcal{K}(K) = \mathcal{K}((K^*K)^{1/2})$, ii) If $\mathcal{K}(K) = \{0\}$, then $\mathcal{D}((K^*K)^{-1})$ is dense in E_1, $\mathcal{D}((K^*K)^{-1}) \subset \mathcal{D}((K^*K)^{-1/2})$ and the closure of $K(K^*K)^{-1/2}$ is an isometric operator $S \in L(E_1, E_2)$, iii) If $\mathcal{K}(K) = \{0\}$ and $x \in \mathcal{D}((K^*K)^{-1/2})$, then $K^*S(K^*K)^{-1/2}x = x$.*

Proof: Let $D = K^*K$.

i) This statement follows from Problem 2.33, §7, Ch. VI of [11].

ii) $\mathcal{K}(D) = \mathcal{K}(K) = \{0\}$. Since D is selfadjoint it follows that $\mathcal{D}(D^{-1})$ is dense in E_1. Using the spectral resolution of D (cf. Ch XI §12 [16]), we obtain $\mathcal{D}(D^{-1}) \subset \mathcal{D}(D^{-1/2})$. Let $x \in \mathcal{D}(D^{-1/2}) \cap \mathcal{D}(D^{1/2})$. Then $\|KD^{-1/2}x\|^2_{E_2} = (D^{-1/2}x, K^*KD^{-1/2}x)_{E_2} = \|x\|^2_{E_1}$. Since $\mathcal{D}(D^{-1/2}) \cap \mathcal{D}(D^{1/2})$ is dense in E_1, it now follows that the closure S is an isometric operator.

iii) Let $x \in \mathcal{D}(D^{-1})$. Then $D^{-1/2}x \in \mathcal{D}(D^{-1/2})$ and $S = KD^{-1/2}$ on $\mathcal{D}(D^{-1/2})$, so $K^*SD^{-1/2}x = K^*KD^{-1}x = x$. This equality extends by continuity to $x \in \mathcal{D}(D^{-1/2})$. **QED**

The spectrum $\sigma(K)$, of a compact selfadjoint K operator on E, is real, denumerable and zero is the only possible accumulation point. The spectral resolution of K is given by

$$K = \sum_{\lambda \in \sigma(K)} \lambda \, e_K(\{\lambda\}), \tag{A.2}$$

where e_K is the corresponding resolution of the identity defined on the Borel subsets of $\mathbb{R}$. If f is a real valued function on $\mathbb{R}$, then the operator $f(K)$ in E is given by

$$f(K) = \sum_{\lambda \in \sigma(K)} f(\lambda)\, e_K(\{\lambda\}), \tag{A.3}$$

on its domain $\mathcal{D}(f(K)) = \{x \in E \mid \sum_{\lambda\in\sigma(K)} |f(\lambda)|^2 \, \|e_K(\{\lambda\})x\|_E^2 < \infty\}$.

Lemma A.2 *Let A be the set of compact selfadjoint operators in E, endowed with the operator norm topology. If $f : \mathbb{R} \to \mathbb{R}$ is a Borel function, bounded on bounded subsets of $\mathbb{R}$, then the function $A \times E \ni (K, x) \mapsto f(K)x \in E$ is Borel measurable. Moreover the mapping $\mathbb{R} \times A \times E \ni (\lambda, K, x) \mapsto e_K(\{\lambda\})x \in E$ is Borel measurable.*

Proof: Here $I \in L(E)$ is the identity operator and $L(E)$ is given the operator norm topology. Let $B = \{K - \lambda I \,|\, K \in A,\ \lambda \in \mathbb{R}\}$ be endowed with the operator norm topology. B is a closed subalgebra of $L(E)$. The subspace $A_0 = \{K - \lambda I \in B \,|\, \lambda \neq 0\}$ is open in B. For given $M = K - \lambda I \in A_0$, the space $\mathcal{K}(M)$ has finite dimension, since K is compact, and $\mathcal{R}(M)$ is a closed subspace of E. It now follows as in the finite dimensional case (cf. Chap. 1, Lemma 4.4 of [10]), that the mapping $A_0 \times E \ni (M, x) \mapsto e_M(\{0\})x \in E$ is Borel measurable. Since $(\mathbb{R}-\{0\})\times A \ni (\lambda, K) \mapsto K - \lambda I \in A_0$ is continuous and $e_{K-\lambda I}(\{0\}) = e_K(\{\lambda\})$, the mapping $F : (\mathbb{R} - \{0\}) \times A \times E \to E$, where $F(\lambda, K, x) = e_K(\{\lambda\})x$, is Borel measurable.

Suppose that f satisfies the hypothesis of the lemma and let $G(K, x) = f(K)x$. We first consider the case of $f(x) = 0$ for all $x \leq 0$. For $K \in A$, let $\mu_1(K) \geq \cdots \mu_n(K) \cdots \geq 0$ be the decreasing sequence of positive eigenvalues of K, each repeated a number of times equal to the multiplicity of the eigenvalue. The function $A \ni K \mapsto \mu_n(K)$ is then continuous (cf. [11], Ch. IV, §3.5). Define $\mu_0(K) = \mu_1(K) + 1$ and $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = 0$ if $x \leq y$ and $u(x, y) = 1$ if $x > y$. u is a Borel function. It follows from (A.3) that

$$G(K, x) = \sum_{n=1}^{\infty} u(\mu_{n-1}(K), \mu_n(K)) f(\mu_n(K))\, e_K(\{\mu_n(K)\})x,\ x \in E. \tag{A.4}$$

The mapping $G : A \times E \to E$ is Borel measurable. In fact, by the continuity of μ_n and the measurability of u and F, each term in the sum (A.4) is Borel measurable, as a function of (K, x). The sum (A.4) converges pointwise (K, x) in E to $G(K, x)$, so G is Borel measurable (cf. Theorem 5.6.3 of [15]).

Next we consider the case of $f(x) = 0$ for all $x \geq 0$. Similarly as to the previous case, it follows that G is a Borel function. Finally we consider

the case of $f(x) = 0$ for $x \neq 0$ and $f(0) = a$. From the two previous cases it follows that $(K, x) \mapsto h(K)x$ is Borel measurable, where $h(0) = 0$ and $h(x) = 1$ for $x \neq 0$. Since $G(K, x) = x - ah(K)x$, it follows that G is a Borel function. The case of a general f now follows by the decomposition $f = f_- + f_0 + f_+$, where the support of f_-, f_0 and f_+ is a subset of $]-\infty, 0[$, $\{0\}$ and $]0, \infty[$ respectively.

To prove the last statement, we note that $A \times E \ni (K, x) \mapsto e_K(\{0\})x \in E$ is Borel measurable, which follows from the identity $e_K(\{0\})x = x - h(K)x$, where $h(0) = 1$, $h(\lambda) = 0$ for $\lambda \neq 0$. The measurability of $\mathbb{R} \times A \times E \ni (\lambda, K, x) \mapsto e_K(\{\lambda\})x \in E$ now follows from the Borel measurability of F. **QED**

Lemma A.3 *Let A be the set of compact operators with trivial kernel from E_1 to E_2, endowed with the operator norm topology. If S_K is the closure of the operator $K(K^*K)^{-1/2}$ then the function $A \times E_1 \ni (K, x) \mapsto S_K x \in E_2$ is Borel measurable. Moreover, if $L \in L(E_1)$ and A' is the subspace of elements $K \in A$ such that $\mathcal{R}(L) \subset \mathcal{D}((K^*K)^{-1/2})$, then $A' \times E_1 \ni (K, x) \mapsto (K^*K)^{-1/2}Lx \in E_1$ is Borel measurable.*

Proof: Let $f_n(x) = \sqrt{x}$ if $x \geq 1/n$ and $f_n(x) = 0$ if $x < 1/n$, for $n \in \mathbb{N}^*$. The function $A \ni K \mapsto K^*K \in A$ is continuous and K^*K is selfadjoint. Let $F_n(K, x) = K f_n(K^*K)x$. Lemma A.2 shows that $F_n : A \times E_1 \to E_2$ is Borel measurable. Since $F_n(K, x)$ converges pointwise to $S_K x$ in E_2 as $n \to \infty$, it follows that $(K, x) \mapsto S_K x$ is Borel measurable. To prove the second statement, we note that $A' \times E_1 \ni (K, x) \mapsto f_n(K^*K)Lx \in E_1$ is Borel measurable. Since $Lx \in \mathcal{D}((K^*K)^{-1/2})$, the sequence $f_n(K^*K)Lx$ converges pointwise (K, x) in E_1 to $(K^*K)^{-1/2}Lx$. It follows that $(K, x) \mapsto (K^*K)^{-1/2}Lx$ is Borel measurable. **QED**

We shall define two mappings, g_0 and g_1, on the space of selfadjoint Hilbert-Schmidt operators on E. They will satisfy $g_0(K) \in \mathcal{K}(K)$ and $g_1(K) \in \mathcal{R}(K)^c$. Let $\{u_n\}_{n \geq 1}$ be an orthonormal basis in E and let K be a selfadjoint Hilbert-Schmidt operators on E. We define

$$g_0(K) = 0 \text{ if } \mathcal{K}(K) = \{0\} \text{ and } g_0(K) = \frac{e_K(\{0\})u_{N(K)}}{\|e_K(\{0\})u_{N(K)}\|} \text{ if } \mathcal{K}(K) \neq \{0\}, \tag{A.5}$$

where $N(K) = \min\{n \,|\, e_K(\{0\})u_n \neq 0\}$. If $\lambda \notin \sigma(K)$, then let $h(K, \lambda) = 0$ and if $\lambda \in \sigma(K)$ has multiplicity m, then let

$$h(K, \lambda) = v_1 + \cdots + v_m,$$

where $\{v_1, \cdots, v_m\}$ is the orthonormal basis in $e_K(\{\lambda\})E$ given by the Schmidt orthonormalization of $\{e_K(\{\lambda\})u_n\}_{n \geq 1}$. More precisely let $P_0 = e_K(\{\lambda\})$ and we construct inductively $n_1, \ldots, n_m$, $v_1, \ldots, v_m$ and $P_1, \ldots, P_m$ by:

$n_1 = \min\{n \,|\, P_0 u_n \neq 0\}$, $v_1 = P_0 u_{n_1}/\|P_0 u_{n_1}\|$ and $P_1 x = (v_1, x)_E v_1$. $Q_{k+1} = P_0 - \sum_{i=1}^k P_i$, $n_{k+1} = \min\{n \,|\, Q_{k+1} u_n \neq 0\}$, $v_{k+1} = Q_{k+1} u_{n_{k+1}}/\|Q_{k+1} u_{n_{k+1}}\|$ and $P_{k+1} x = (v_{k+1}, x)_E v_{k+1}$.

We now define g_1 by

$$g_1(K) = g_1'(K)/\|g_1'(K)\|, \text{ where } g_1'(K) = \sum_{\lambda \in \sigma(K)} \lambda h(K, \lambda) + g_0(K). \quad \text{(A.6)}$$

Lemma A.4 *Let A be the set of selfadjoint Hilbert-Schmidt operators on E, endowed with the operator norm topology. The maps $g_i : A \to E$, $i = 0, 1$ given by (A.5) and (A.6) are Borel measurable. For every $K \in A$ the following two properties are satisfied: i) $g_0(K) \in \mathcal{K}(K)$ and if $\mathcal{K}(K) \neq 0$ then $\|g_0(K)\| = 1$. ii) $g_1(K) \notin \mathcal{R}(K)$ and $\|g_1(K)\| = 1$.*

Proof: Since Hilbert-Schmidt operators are compact it follows from Lemma A.2 that $\mathbb{R} \times A \times E \ni (\lambda, K, x) \mapsto e_K(\{\lambda\})x \in E$ is Borel measurable. The measurability of g_0 then follows from that N is measurable and that $x \mapsto x/\|x\|$ is measurable on $E - \{0\}$. Similarly, for given (K, λ), v_i is a measurable function of a finite number of the variables $e_K(\{\lambda\})u_n$. Therefore $h : A \times \mathbb{R} \to E$ is measurable. The sum in (A.6) converges. In fact, using that $\|h(K, \lambda)\|_E^2$ is equal to the dimension of $e_K(\{\lambda\})E$, it follows that

$$\|g_1'(K) - g_0'(K)\|_E^2 = \sum_{\lambda \in \sigma(K)} \lambda^2 \|h(K, \lambda)\|_E^2 = \|K\|_{H-S}^2. \quad \text{(A.7)}$$

The Borel measurability of g_1 follows from the pointwise convergence.

Statement *i*) is obvious and we prove *ii*). If $\mathcal{K}(K) \neq \{0\}$, then $e_K(\{0\})g_1(K) = g_0(K) \neq 0$ and $e_K(\{0\})\mathcal{R}(K) = \{0\}$, show that $g_1'(K) \notin \mathcal{R}(K)$. Let $\mathcal{K}(K) = \{0\}$, suppose that $g_1'(K) \in \mathcal{R}(K)$, let $x \in E$ be the unique element such that $g_1'(K) = Kx$ and let $x_\lambda = e_K(\{\lambda\})x$. Then $Kx_\lambda = e_K(\{\lambda\})Kx = Kh(K, \lambda)$, so $x_\lambda = h(K, \lambda)$. This gives that $\|x\|_E^2 = \sum_{\lambda \in \sigma(K)} \|h(K, \lambda)\|_E^2 = \infty$. This is a contradiction, so $g_1'(K) \notin \mathcal{R}(K)$. Hence $g_1'(K) \notin \mathcal{R}(K)$ for every $K \in A$. In particular $g_1'(K) \neq 0$, for every $K \in A$, so $g_1(K)$ is well-defined and $\|g_1(K)\| = 1$. **QED**

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